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Chiral Symmetry and Quark Confinement

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Abstract

In the physical vacuum of QCD, the energy density of light-quark fields strongly coupled to slowly varying gluon fields can be negative, and so a condensate of pairs of quarks and antiquarks of nearly opposite momenta forms which breaks chiral symmetry, confines quarks, and makes gluons massive.

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1 The QCD Vacuum

The thesis of this note is that chiral-symmetry breaking, quark confinement, and the short range of the strong force all arise from the same feature of QCD, namely that the energy density of strongly coupled light [1] quarks can be negative. The hamiltonian H_q of the u, d, and s quarks

$$H_q = \sum_{f=u,d,s} \int d^3x \, \bar{\psi}_f \left(\vec{\gamma} \cdot \vec{\nabla} - ig\gamma^0 A_{0a} \frac{\lambda_a}{2} - ig\vec{\gamma} \cdot \vec{A}_a \frac{\lambda_a}{2} + m_f \right) \psi_f \tag{1}$$

can assume large negative mean values due to the term $-g \int d^3x \vec{J}_a \cdot \vec{A}_a$ when the gauge field $\vec{A}_a$ varies slowly with a modulus $|\vec{A}_a|$ that exceeds m_u/g by a sufficient margin [2]. For nearly constant gauge fields $\vec{A}_a$, the states that drive the energy lowest are condensates [3] of pairs of light quarks and antiquarks of opposite momenta; in such pairs the color charges cancel, but the color currents add. When $g|\vec{A}_a| >> m_d$, the u and d quarks play very similar roles, but pairs of s quarks and antiquarks are important only when $g|\vec{A}_a| \gg m_s$.

If the gauge fields are not only slowly varying but also essentially abelian, in the sense that $gf_{abc}A^b_\mu A^c_\nu$ is small (*e.g.*, because $A^a_\mu(x) \simeq C^a(x)V_\mu(x)$), then the energy of the gauge fields is also small. If an essentially abelian gauge field, *e.g.,* $|\vec{A}_8|$, is relatively constant over a sphere of radius R beyond which it either remains constant or slowly drops to zero, then its energy density near the sphere can be of the order of $|\vec{A}_8|^2/R^2$ or less while that of the light-quark condensate can be as negative as $-|g\vec{A}_8|^4$. The physical vacuum of QCD is therefore a linear combination of states, each of which is approximately a coherent [4] state $|\vec{A}\rangle$ of a slowly varying, essentially abelian gauge field $\vec{A}_a(x)$ and an associated condensate of pairs of u and d quarks and $\bar{u}$ and $\bar{d}$ antiquarks of nearly opposite momenta:

$$|\Omega\rangle \simeq \int D\vec{A}_a \, f(\vec{A}_a) \prod_{S(A,u)} a^\dagger(\vec{p}_i, \sigma, u_i) a^{c\dagger}(\vec{q}_j, \tau, u_j) \prod_{S(A,d)} a^\dagger(\vec{p}_i, \sigma, d_i) a^{c\dagger}(\vec{q}_j, \tau, d_j) |\vec{A}\rangle. \tag{2}$$

Here the sets $S(A,u)$ and $S(A,d)$ specify the momenta $\vec{p}, \vec{q}$, spins σ, τ, and colors i, j of the quarks and antiquarks, the operator $a^{c\dagger}(\vec{q}, \tau, d_j)$ creates a d antiquark of momentum $\vec{q}$, spin τ, and color j, the function $f(\vec{A}_a)$ is a weight function, and the s quarks have been suppressed.

In what follows I shall compute the energy density of a such a light-quark condensate for the case of a constant gauge field $\vec{A}_8$. It will turn out that if $g|\vec{A}_8|$ is of the order of a GeV, then the mean value $\langle \frac{1}{2}(\bar{u}u + \bar{d}d)\rangle$ of the light-quark condensate is about $(260\,\mathrm{MeV})^3$ as required by soft-pion physics.

Because a color-electric field moves quarks in one direction and antiquarks in the opposite direction, the quark condensate of the QCD vacuum in this model is not stable in the presence of color-electric fields. Thus volumes of space that are traversed by color-electric fields have less quark-antiquark condensate and hence a higher energy density than that of the physical vacuum. Consequently the surface of a hadron is exposed to a pressure that is equal to the difference between the energy density ρ_Ω of the physical vacuum outside the hadron and the energy density ρ_h inside the hadron which, due to the color-electric fields, is somewhat higher than ρ_Ω. This pressure p

$$p \simeq \rho_h - \rho_\Omega \tag{3}$$

confines quarks because it squeezes their color-electric fields. Thus quarks are confined not because of the energy of their color-electric fields but because their color-electric fields are excluded by the physical vacuum. In the example which follows, a very small decrease in the quark-antiquark condensate results in a pressure p of the order of $(1\,\mathrm{GeV})^4$.

2 A Particular Condensate

Let us consider the case of a constant gauge field $\vec{A}_8$ that points in the direction 8 of color space; the energy density and quark condensate associated with a slowly varying gauge field that points in an arbitrary direction in color space should be similar. If we call the quark colors red, green, and blue, then the condensate will be made of red and green u, d, and s quarks of momentum $\vec{p}$ and both spin indices σ; red and green u, d, and s antiquarks of momentum $-\vec{p}$ and both spin indices σ; blue u, d, and s quarks of momentum $-\vec{p}$ and both spin indices σ; and blue u, d, and s antiquarks of momentum $\vec{p}$ and both spin indices σ. The domains of integration $S(A,u)$ and $S(A,d)$ for the u and d quarks will be very similar when the gauge field $\vec{A}_8$ is intense, but the domain for the s quarks will be smaller. The component $|\Omega_A\rangle$ of the

QCD vacuum associated with the gauge field $\vec{A}_8$ will then be

$$|\Omega_A\rangle = \prod_{S(A,u)} a^\dagger(\vec{p}_i,\sigma,u_i)a^{c\dagger}(-\vec{p}_i,\sigma,u_i) \prod_{S(A,d)} a^\dagger(\vec{p}_i,\sigma,d_i)a^{c\dagger}(-\vec{p}_i,\sigma,d_i)\,|\vec{A}_8\rangle \tag{4}$$

apart from the s quarks. These products over momentum, spin, and color are defined by box quantization in a volume V, and so the mean value of the hamiltonian H_q in the state $|\Omega_A\rangle$ is really an energy density.

The only quark operators that have non-zero mean values in the state $|\Omega_A\rangle$ are those that destroy and create the same kind of quark or antiquark. Thus if we normally order the quark hamiltonian (1), then the part of the magnetic term $H_{qm} = -g\int d^3x\,\vec{J}_a\cdot\vec{A}_a$ that involves the field $\psi_d(x)$ of the d quark

$$\psi_{\ell d}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}\left[u_\ell(\vec{p},\sigma,d_i)e^{ip\cdot x}a(\vec{p},\sigma,d_i) + v_\ell(\vec{p},\sigma,d_i)e^{-ip\cdot x}a^{c\dagger}(\vec{p},\sigma,d_i)\right] \tag{5}$$

has a mean value

$$\begin{aligned} E_{dm} &= \langle\Omega_A|H_{dm}|\Omega_A\rangle = \langle\Omega_A|\left(-g\int d^3x\,\vec{J}^d_a\cdot\vec{A}_a\right)|\Omega_A\rangle \\ &= \langle\Omega_A|\left(-ig\int d^3x\,\bar{\psi}_d\vec{\gamma}\cdot\vec{A}_a\frac{\lambda_a}{2}\psi_d\right)|\Omega_A\rangle \end{aligned} \tag{6}$$

given by

$$\begin{aligned} E_{dm} &= g\vec{A}_8\cdot\sum_{\sigma,i}\frac{\lambda^8_{ii}}{2}\int_{S(A,d)} d^3p\,\left[u^\dagger(\vec{p}_i,\sigma,d_i)\gamma^0\vec{\gamma}\,u(\vec{p}_i,\sigma,d_i)\right. \\ &\qquad \left. -v^\dagger(-\vec{p}_i,\sigma,d_i)\gamma^0\vec{\gamma}\,v(-\vec{p}_i,\sigma,d_i)\right]. \end{aligned} \tag{7}$$

The spin sums [5]

$$\sum_\sigma u_\ell(\vec{p},\sigma,d_i)u^*_{\ell'}(\vec{p},\sigma,d_i) = \frac{1}{2p^0}\left[\left(p^\mu\gamma_\mu + im_d\right)\gamma^0\right]_{\ell\ell'} \tag{8}$$

and

$$\sum_\sigma v_\ell(\vec{p},\sigma,d_i)v^*_{\ell'}(\vec{p},\sigma,d_i) = \frac{1}{2p^0}\left[\left(p^\mu\gamma_\mu - im_d\right)\gamma^0\right]_{\ell\ell'} \tag{9}$$

imply the trace relations

$$u^\dagger(\vec{p}_i,\sigma,d_i)\gamma^0\vec{\gamma}\,u(\vec{p}_i,\sigma,d_i) = v^\dagger(\vec{p}_i,\sigma,d_i)\gamma^0\vec{\gamma}\,v(\vec{p}_i,\sigma,d_i) = -\frac{2\vec{p}}{p^0} \tag{10}$$

and

$$u^\dagger(\vec{p}_i,\sigma,d_i)\,u(\vec{p}_i,\sigma,d_i) = v^\dagger(\vec{p}_i,\sigma,d_i)\,v(\vec{p}_i,\sigma,d_i) = 2. \tag{11}$$

Thus the magnetic energy density of the d quarks and antiquarks of color i in the constant gauge field $\vec{A}_8$ is

$$E_{dm} = -2g\lambda^8_{ii}\int_{S(A,d)}\frac{d^3p}{p^0}\,\vec{A}_8\cdot\vec{p}. \tag{12}$$

The same spin sums imply that in the state $|\Omega_A\rangle$, the mean value of the color charge density $J^{0d}_a = \psi^\dagger_d\frac{1}{2}\lambda_a\psi_d$ and that of the color current $\vec{J}^d_a = -\psi^\dagger_d\gamma^0\vec{\gamma}\frac{1}{2}\lambda_a\psi_d$ for $a \neq 8$ both vanish. Thus the mean value of the second term of the hamiltonian (1) is zero.

The mean value of the hamiltonian H_q for d quarks and antiquarks of color i in the state $|\Omega_A\rangle$ is therefore

$$E_{di} = \int_{S(A,d,i)} d^3p\left(4p^0 - 2g\lambda^8_{ii}\frac{\vec{A}_8\cdot\vec{p}}{p^0}\right) \tag{13}$$

where the domain of integration $S(\vec{A}_8,d,i)$ is the set of momenta $\vec{p}$ for which the integrand is negative

$$g\lambda^8_{ii}\vec{A}_8\cdot\vec{p} > 2(\vec{p}^2 + m_d^2). \tag{14}$$

The set $S(\vec{A}_8,d,i)$ is empty unless $g\lambda^8_{ii}|\vec{A}_8| > 4m_d$, which requires the effective magnitude of the gauge field to be large compared to the mass m_d of the d quark.

Since the quark energy density E_{fi} depends upon the flavor f and the color i only through the dimensionless ratio

$$r = \frac{4m_f}{g\lambda^8_{ii}|\vec{A}_8|}, \tag{15}$$

we may write it as the integral

$$E_{fi} = -32\pi(g\lambda^8_{ii}|\vec{A}_8|)^4 \int_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}} dx\, \frac{x(2x - x^2 - r^2)^2}{\sqrt{x^2+r^2}} \tag{16}$$

which has the value

$$\begin{aligned} E_{fi} &= -\,32\pi(g\lambda^8_{ii}|\vec{A}_8|)^4 \\ &\quad \left[\left(\frac{1}{5}(x^2+r^2)^2 - x^3 - \frac{1}{2}r^2 x + \frac{4}{3}(x^2 - 2r^2)\right)\sqrt{x^2+r^2}\right. \\ &\qquad \left. + \frac{1}{2}r^4 \operatorname{arcsinh}\left(\frac{x}{r}\right)\right]_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}} . \end{aligned} \tag{17}$$

For small r the energy density E_{fi} is approximately

$$E_{fi} \simeq -32\pi(g\lambda^8_{ii}|\vec{A}_8|)^4 \left(\frac{16}{15} - 4\left(\frac{4m_f}{g\lambda^8_{ii}|\vec{A}_8|}\right)^2\right). \tag{18}$$

Summing over the three colors, we get

$$E_f \simeq -64\pi(g|\vec{A}_8|)^4 \left(\frac{16}{15} - 4\left(\frac{4m_f}{g|\vec{A}_8|}\right)^2\right) \tag{19}$$

which displays isospin symmetry when $4m_f \ll g|\vec{A}_8|$. If the gauge field is moderately strong $4m_s > g\lambda^8_{ii}|\vec{A}^8| \gg 4m_d$, then the energy density of the u–d condensate is

$$E_{ud} \simeq -128\pi(g|\vec{A}_8|)^4 \left(\frac{16}{15} - 4\left(\frac{4m_{ud}}{g|\vec{A}_8|}\right)^2\right) \tag{20}$$

where $m^2_{ud} = (m^2_u + m^2_d)/2$. For stronger gauge fields, $g\lambda^8_{ii}|\vec{A}^8| >> 4m_s$, the energy density of the light-quark condensate is

$$E_{uds} \simeq -192\pi(g|\vec{A}_8|)^4 \left(\frac{16}{15} - 4\left(\frac{4m_\ell}{g|\vec{A}_8|}\right)^2\right) \tag{21}$$

where $m^2_\ell = (m^2_u + m^2_d + m^2_s)/3$.

3 The Breakdown of Chiral Symmetry

The quark condensate occasioned by the constant gauge field $\vec{A}_8$ gives rise to a mean value of the space average of $\frac{1}{2}(\bar{u}u+\bar{d}d)$, which is an order parameter that traces the breakdown of chiral symmetry. By using the spin sums (8) and (9) and the expansion (5) of the Dirac field, we find for this order parameter

$$\begin{aligned}\langle\tfrac{1}{2}(\bar{u}u+\bar{d}d)\rangle &= \langle\Omega_A|\int d^3x\tfrac{1}{2}(\bar{u}u+\bar{d}d)|\Omega_A\rangle \\ &= \tfrac{1}{2}\sum_{f=u}^{d}\sum_{i=1}^{3}\sum_{\sigma}\int_{S(A,f,i)} d^3p\,\big[u^\dagger(\vec{p}_i,\sigma,f_i)i\gamma^0u(\vec{p}_i,\sigma,f_i) \\ &\qquad -v^\dagger(-\vec{p}_i,\sigma,f_i)i\gamma^0v(-\vec{p}_i,\sigma,d_i)\big] \\ &= \sum_{f=u}^{d}\sum_{i=1}^{3}2m_f\int_{S(A,f,i)}\frac{d^3p}{p^0}. \end{aligned} \tag{22}$$

In terms of the ratio $r=4m_f/(g\lambda^8_{ii}|\vec{A}_8|)$, this order parameter is

$$\begin{aligned}\langle\tfrac{1}{2}(\bar{u}u+\bar{d}d)\rangle &= \sum_{f=u}^{d}\sum_{i=1}^{3}\frac{\pi}{4}m_f\left(g\lambda^8_{ii}|\vec{A}_8|\right)^2\int_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}}dx\left(\frac{x^2}{\sqrt{x^2+r^2}}-\frac{x}{2}\sqrt{x^2+r^2}\right) \\ &= \sum_{f=u}^{d}\sum_{i=1}^{3}\frac{\pi}{4}m_f\left(g\lambda^8_{ii}|\vec{A}_8|\right)^2 \\ &\quad\left[\left(\frac{x}{2}-\frac{x^2+r^2}{6}\right)\sqrt{x^2+r^2}-\frac{r^2}{2}\operatorname{arcsinh}\left(\frac{x}{r}\right)\right]_{1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}}. \end{aligned} \tag{23}$$

For small r this condensate or order parameter is

$$\langle\tfrac{1}{2}(\bar{u}u+\bar{d}d)\rangle \simeq \sum_{f=u}^{d}\sum_{i=1}^{3}\frac{\pi}{4}m_f\left(g\lambda^8_{ii}|\vec{A}_8|\right)^2\left[\frac{2}{3}+\frac{r^2}{2}\left(\ln\left(\frac{r}{4}\right)-\frac{1}{2}\right)\right]. \tag{24}$$

In the limit $r\to 0$ and summed over colors (and over u and d), it is

$$\langle\tfrac{1}{2}(\bar{u}u+\bar{d}d)\rangle \simeq \frac{\pi}{3}(m_u+m_d)\left(g|\vec{A}_8|\right)^2. \tag{25}$$

We may use this formula and Weinberg's relation (Eq.(19.4.46) of [5])

$$\langle \tfrac{1}{2}(\bar{u}u + \bar{d}d)\rangle \simeq \frac{m_\pi^2 F_\pi^2}{4(m_u + m_d)} \tag{26}$$

in which $F_\pi \simeq 184\,\mathrm{MeV}$ is the pion decay constant, to estimate the effective magnitude $\langle g|\vec{A}_8|\rangle$ of the gauge field in the physical vacuum of QCD as

$$\langle g|\vec{A}_8|\rangle \simeq \sqrt{\frac{3}{4\pi}}\,\frac{m_\pi F_\pi}{(m_u + m_d)}. \tag{27}$$

In the $\overline{\mathrm{MS}}$ scheme at a renormalization scale $\mu = 1\,\mathrm{GeV}$, the mass range $3\,\mathrm{MeV} < (m_u + m_d)/2 < 8\,\mathrm{MeV}$ of the Particle Data Group [6] implies that $770\,\mathrm{MeV} < \langle g|\vec{A}_8|\rangle < 2070\,\mathrm{MeV}$. Since $\mu = 1\,\mathrm{GeV}$ is somewhat high for hadronic physics, the low end of the range, $\langle g|\vec{A}_8|\rangle \simeq 800\,\mathrm{MeV}$, may be more reliable. With this estimate of $\langle g|\vec{A}_8|\rangle$, a drop of only 1% in the hadronic quark-antiquark condensate would produce a confining pressure of the order of $p \simeq (1\mathrm{GeV})^4$.

4 The Range of the Strong Force

Let us now specialize to the component $|\Omega_A\rangle$ of the QCD vacuum that has a constant gauge field A_8^1 pointing in direction 1 of space and direction 8 of color space. In this component the mean value of the hamiltonian of the gauge fields contains a mass term for the fields A_b^m for $m \neq 1$

$$\sum_{a,b,c,m} \tfrac{1}{2} f_{a8b} f_{a8c} \langle A_8^1\rangle^2 A_b^m A_c^m = \sum_{b,c,m} \tfrac{1}{2} M_{bc}^2 A_b^m A_c^m \tag{28}$$

in which the mass matrix is

$$M_{bc}^2 = \langle gA_8^1\rangle^2 \sum_a f_{a8b} f_{a8c} = \langle gA_8^1\rangle^2 \left(T_8^2\right)_{bc} \tag{29}$$

where the 8×8 matrix T_8 is a generator of the group *SU(3)* in the adjoint representation. Since the matrix T_8 is hermitian, the eigenvalues of the mass matrix M_{bc}^2 are all non-negative.

The QCD vacuum (2) is an integral over slowly varying gauge fields and their correlated condensates. In this vacuum $|\Omega\rangle$, every space component A_c^i of the gauge field acquires a non-zero mean value. Thus gluons are massive in the vacuum of QCD, and according to the estimate (27), the mass of the gluon is in the range of hundreds of MeV. This mass explains why the strong force is of short range and why strong van der Waals forces are not seen in low-energy hadronic scattering even though gluons are massless in perturbation theory. But to the extent that color-electric fields sweep away the quark-antiquark condensate, gluons are massless inside hadrons, particularly near the quarks.

5 Summary and Conclusions

We have seen that a vacuum component $|\Omega_A\rangle$ consisting of a coherent state of a slowly varying gauge field $\vec{A}_c$ and an associated quark-antiquark condensate (4) possesses a negative energy density ρ_Ω of the order of $-(g|\vec{A}_c|)^4$. The difference between ρ_Ω and the energy density ρ_h inside hadrons confines quarks. The $q\bar{q}$ condensate in the component $|\Omega_A\rangle$ leads to a spontaneous breaking of chiral symmetry with an order parameter $\langle(\bar{u}u + \bar{d}d)/2\rangle$ that agrees with soft-pion physics if the effective strength $g|\vec{A}_c|$ of the gauge field is of the order of 800 MeV. If the gauge field $\vec{A}_c$ points in the direction 1, then some of the gauge fields A_a^m for $m \neq 1$ become massive.

The real vacuum is an integral over all such components $|\Omega_A\rangle$. In the temporal gauge, the gauge field A_a^0 is absent, and the vacuum is an integral (2) over all gauge transformations $\omega(\vec{x})$ of the image $|\Omega_A^\omega\rangle$ of a state like the component $|\Omega_A\rangle$ under the gauge transformation $\omega(\vec{x})$. This integral removes any breaking of rotational invariance associated with the uniform field A_8^1. The mean value of any gauge-invariant operator, for instance the quark hamiltonian H_q, is a double integral

$$\langle\Omega|H_q|\Omega\rangle = \int D\vec{A}_a \int D\vec{A}'_a \, \langle\Omega_A|H_q|\Omega_{A'}\rangle \tag{30}$$

in which most of the off-diagonal terms are very small. In the vacuum $|\Omega\rangle$, all the gluons acquire masses in the range of hundreds of MeV.

Of course, the actual energy density of the vacuum is small and non-negative. But by using normal ordering, we have been ignoring the zero-point energies of the fields. Zero-point energies augment ρ_Ω by a positive or negative energy density that is quartically divergent unless the number of Fermi fields is equal to the number of Bose fields and is quadratically divergent unless the super-trace $\sum(-1)^{2j} m_j^2$ of the squared masses of all particles vanishes. The large negative energy density ρ_Ω may make it possible to cancel a large positive energy density due to the breaking of supersymmetry.

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[3] The quark-antiquark condensates considered here should not be confused with the diquark condensates studied by M. Alford, K. Rajagopal, and F. Wilczek, hep-ph/9802284 and hep-ph/9804403, and by T. Schäfer and F. Wilczek, hep-ph/9811473.

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